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A Lagrangian Formalism Based on the Velocity-Determined Virtual Displacements for Systems with Nonholonomic Constraints

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Abstract We develop a Lagrangian formulation for classical systems with a general nonholonomic constraints by utilizing the so-called velocity-determined virtual-displacement conditions, i.e. by assuming the virtual displacements to be along the direction of the velocities in a special reference frame. It is shown that our general scheme encompasses as special cases the Chetaev and Voronets approaches when the constraints are homogeneous or linear in relative velocities.

Keywords Constraint · Nonholonomic system · Vakonomic model · Chetaev model

1 Introduction

Nonholonomic mechanics is central for the treatment of a number of important physical processes in robotics, wheeled dynamics and motion generation (see e.g., the review articles [1, 2] and references therein). The mechanics of systems with nonholonomic constraints is inherently different from that with holonomic ones: for *holonomic* constraints one conventionally eliminates the dependent coordinates by exploiting the constraint conditions. This yields a set of independent coordinates, the so-called generalized coordinates [3, 4], in terms of which one expresses the equations of motion. Unfortunately, this standard scheme is not applicable directly to systems with *noholonomic* constraints because dependent coordinates cannot be solved for inversely from the noholonomic constraint equations. Instead, some dependent velocities can be solved for. It is important to recall that the various theoretical approaches to nonholonomic mechanics are restricted by the functional dependencies of the

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constraints: for systems which has *cyclic coordinates* (in the sense defined below), Chaplygin [5] derived an equation of motion by using the variational principle. Subsequently, Chaplygin's equation was generalized to constraints that are linear in velocities [6] and to the constraints for which the Chetaev condition [7–9] holds. On the other hand, the Chetaev condition *does not* hold for all nonholonomic constraints, except for those which are homogeneous in relative velocities [10, 11]. For a more detailed review of nonholonomic mechanics we refer to the overview articles [1, 2]. A Hamiltonian-based treatment is exposed in [12] whereas the method of [1] rests on a (geometric) Lagrangian formalism. The connections between these approaches have been established in [2]. In the context of this present work we stress, however that these works [1, 2, 12] apply to the nonholonomic constraints which are *homogeneous or linear* in relative velocities [2].

For a formulation applicable for more general nonholonomic constraints we expand in this work each constraint in a Taylor series in relative velocities, and then make use of the condition for so-called velocity-determined virtual displacements. This amounts to assume the virtual displacements to be along the direction of the relative velocity. We then derive an equation of motion with Lagrange multipliers (the Routh equation) [10]. The use of multipliers implies however certain disadvantages: it increases the number of equations and makes the constraint problem much more complicated. To describe the motion with less variables a multiplier-free Lagrange equation is highly desirable, a task tackled in this paper. We derive a general, multiplier-free form of the equations of motion by using the constraint condition to solve for some independent velocities. The solved dependent velocities are embedded in the expression for the kinetic energy or the Lagrangian. While our derivation is valid for general nonholonomic constraints we show explicitly that it encompasses, as special cases, well-known results:

(a) The equations of motion for the Chetaev system is obtained if the constraints are homogeneous in relative velocities.
(b) The Voronets' equation follows if the constraints are linear in relative velocities.

In what follow we adopt the standard notation of [17] and operate, as done in the standard literature [5, 17] with generalized coordinates. The formulation and the connection of the present theory to the geometric approach of [1] and the Hamiltonian treatment of [12] are the subject of ongoing research.

The paper is organized as follows: in Sect. 2 we derive the equations of motion using a variational approach and some special cases are investigated. As an application of the formulation, an illustrative example of the rolling disk is discussed in Sect. 3. Section 4 summarizes the main findings.

2 The Equation of Motion Based on the Velocity-Determined Virtual Displacements

We consider a classical mechanical system consisting of n particles with masses $m_1, \ldots, m_N$ that are subjected to the following nonholonomic, coordinate (q_α) and velocity ($\dot{q}_\alpha$) dependent constraints

$$f_j(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t) = 0 \quad (j = 1, \ldots, k). \tag{1}$$

Since we have k constraints at hand we may solve inversely for k generalized velocities and write

$$\dot{q}_{s+\beta} = \dot{q}_{s+\beta}(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_s, t) \quad (\beta = 1, \ldots, k), \tag{2}$$

where $s=n-k$ is the number of independent velocities. Exploiting (2), the kinetic energy T is expressible as a function of the n independent coordinates and the s independent velocities, i.e. we embed the constraints in the kinetic energy term [13]

$$\begin{aligned} T &= T(q_1,\dots,q_n,\dot{q}_1,\dots,\dot{q}_s,\dot{q}_{s+1}(q_1,\dots,q_n,\dot{q}_1,\dots,\dot{q}_s,t),\ \dots,\\ &\qquad \dot{q}_n(q_1,\dots,q_n,\dot{q}_1,\dots,\dot{q}_s,t),t)\\ &\equiv \tilde{T}(q_1,\dots,q_n,\dot{q}_1,\dots,\dot{q}_s,t). \end{aligned} \tag{3}$$

The next step consists of expanding (1) or (2) in a Taylor series with respect to the velocities [10]:

$$\dot{q}'_\alpha = \dot{q}_\alpha - v_\alpha^{(0)} \quad (\alpha = 1,\dots,n), \tag{4}$$

where $v_\alpha^{(0)}$ is the velocity of the reference frame. Equation (3) leads to the following condition for virtual displacements:

$$\delta q_{s+\beta} = \sum_{\alpha=1}^{s} B_{s+\beta,\alpha}\delta q_\alpha \quad (\beta = 1,\dots,k), \tag{5}$$

where

$$\begin{aligned} B_{s+\beta,\alpha} &= B_{s+\beta,\alpha}(q_1,\dots,q_n,\dot{q}_1,\dots,\dot{q}_s,t)\\ &= \left(\frac{\partial \dot{q}_{s+\beta}}{\partial \dot{q}_\alpha}\right)_{\dot{q}_1=v_1^{(0)},\dots,\dot{q}_s=v_s^{(0)}} + \frac{1}{2!}\sum_{\alpha_1=1}^{s}\left(\frac{\partial^2 \dot{q}_{s+\beta}}{\partial \dot{q}_\alpha \partial \dot{q}_{\alpha_1}}\right)_{\dot{q}_1=v_1^{(0)},\dots,\dot{q}_s=v_s^{(0)}} \dot{q}'_{\alpha_1}\\ &\quad + \frac{1}{3!}\sum_{\alpha_1,\alpha_2=1}^{s}\left(\frac{\partial^3 \dot{q}_{s+\beta}}{\partial \dot{q}_\alpha \partial \dot{q}_{\alpha_1}\partial \dot{q}_{\alpha_2}}\right)_{\dot{q}_1=v_1^{(0)},\dots,\dot{q}_s=v_s^{(0)}} \dot{q}'_{\alpha_1}\dot{q}'_{\alpha_2} + \cdots\\ &= \sum_{l=0}^{\infty}\frac{1}{(l+1)!}\sum_{\alpha_1,\dots,\alpha_l=1}^{s}\left(\frac{\partial^{l+1}\dot{q}_{s+\beta}}{\partial \dot{q}_\alpha \partial \dot{q}_{\alpha_1}\cdots\partial \dot{q}_{\alpha_l}}\right)_{\dot{q}=0}\dot{q}'_{\alpha_1}\cdots\dot{q}'_{\alpha_l}. \end{aligned} \tag{6}$$

For ideal constraints in which the relative velocities are normal to the constraint forces, we may choose the virtual displacements (called the velocity-determined virtual displacements) parallel to the relative velocities, thus the virtual work of reaction forces on these virtual displacements vanishes and the D'Alembert's equation reads

$$\int_{t_1}^{t_2}\sum_{i=1}^{N}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{\mathbf{r}}_i} - \mathbf{F}_i\right)\cdot \delta\mathbf{r}_i dt = 0 \quad (\delta\mathbf{r}_1|_{t=t_1,t_2}=0,\dots,\delta\mathbf{r}_N|_{t=t_1,t_2}=0), \tag{7}$$

where $\mathbf{F}_i$ is the applied force. In the generalized coordinates (7) becomes

$$\begin{aligned} \int_{t_1}^{t_2}\Bigg[\sum_{\alpha=1}^{s}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\alpha} - \frac{\partial T}{\partial q_\alpha} - Q_\alpha\right)\delta q_\alpha + \sum_{\beta=1}^{k}\Bigg(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_{s+\beta}}\\ - \frac{\partial T}{\partial q_{s+\beta}} - Q_{s+\beta}\Bigg)\delta q_{s+\beta}\Bigg]dt = 0 \quad (\delta q_1|_{t=t_1,t_2}=0,\dots,\delta q_s|_{t=t_1,t_2}=0), \end{aligned} \tag{8}$$

where Q_α is the generalized applied force. With the help of (5) we infer that

$$\int_{t_1}^{t_2}\sum_{\alpha=1}^{s}\Bigg[\frac{d}{dt}\frac{\partial T}{\partial\dot q_\alpha}-\frac{\partial T}{\partial q_\alpha}+\sum_{\beta=1}^{k}\bigg(\frac{d}{dt}\frac{\partial T}{\partial\dot q_{s+\beta}}-\frac{\partial T}{\partial q_{s+\beta}}\bigg)B_{s+\beta,\alpha}$$
$$-Q_\alpha-\sum_{\beta=1}^{k}Q_{s+\beta}B_{s+\beta,\alpha}\Bigg]\delta q_\alpha dt=0. \tag{9}$$

Noting that $\delta q_1,\dots,\delta q_s$ are independent we conclude that

$$\frac{d}{dt}\frac{\partial T}{\partial\dot q_\alpha}-\frac{\partial T}{\partial q_\alpha}+\sum_{\beta=1}^{k}\bigg(\frac{d}{dt}\frac{\partial T}{\partial\dot q_{s+\beta}}-\frac{\partial T}{\partial q_{s+\beta}}\bigg)B_{s+\beta,\alpha}$$
$$=Q_\alpha+\sum_{\beta=1}^{k}Q_{s+\beta}B_{s+\beta,\alpha}\quad(\alpha=1,\dots,s). \tag{10}$$

To obtain the equation of motion for independent coordinates we find at first by exploiting (3) that

$$\frac{d}{dt}\frac{\partial\tilde T}{\partial\dot q_\alpha}=\frac{d}{dt}\Bigg(\frac{\partial T}{\partial\dot q_\alpha}+\sum_{\beta=1}^{k}\frac{\partial T}{\partial\dot q_{s+\beta}}\frac{\partial\dot q_{s+\beta}}{\partial\dot q_\alpha}\Bigg), \tag{11}$$

$$\frac{\partial\tilde T}{\partial q_\alpha}=\frac{\partial T}{\partial q_\alpha}+\sum_{\beta=1}^{k}\frac{\partial T}{\partial\dot q_{s+\beta}}\frac{\partial\dot q_{s+\beta}}{\partial q_\alpha}, \tag{12}$$

$$\frac{\partial\tilde T}{\partial q_{s+\beta}}=\frac{\partial T}{\partial q_{s+\beta}}+\sum_{\gamma=1}^{k}\frac{\partial T}{\partial\dot q_{s+\gamma}}\frac{\partial\dot q_{s+\gamma}}{\partial q_{s+\beta}}. \tag{13}$$

Equations (11) and (12) lead to

$$\frac{d}{dt}\frac{\partial T}{\partial\dot q_\alpha}-\frac{\partial T}{\partial q_\alpha}=\frac{d}{dt}\frac{\partial\tilde T}{\partial\dot q_\alpha}-\frac{\partial\tilde T}{\partial q_\alpha}-\sum_{\beta=1}^{k}\Bigg[\bigg(\frac{d}{dt}\frac{\partial T}{\partial\dot q_{s+\beta}}\bigg)\frac{\partial\dot q_{s+\beta}}{\partial\dot q_\alpha}$$
$$+\frac{\partial T}{\partial\dot q_{s+\beta}}\frac{d}{dt}\bigg(\frac{\partial\dot q_{s+\beta}}{\partial\dot q_\alpha}\bigg)-\frac{\partial T}{\partial\dot q_{s+\beta}}\frac{\partial\dot q_{s+\beta}}{\partial q_\alpha}\Bigg]. \tag{14}$$

Inserting (14) and (13) into (10) and using the notation

$$Q_\alpha+\sum_{\beta=1}^{k}Q_{s+\beta}B_{s+\beta,\alpha}\equiv\tilde Q_\alpha(q_1,\dots,q_n,\dot q_1,\dots,\dot q_s,t)\quad(\alpha=1,\dots,s), \tag{15}$$

we obtain the following equation of motion for the ideal nonholonomic systems

$$\frac{d}{dt}\frac{\partial\tilde T}{\partial\dot q_\alpha}-\frac{\partial\tilde T}{\partial q_\alpha}-\sum_{\beta=1}^{k}\frac{\partial\tilde T}{\partial q_{s+\beta}}B_{s+\beta,\alpha}+\sum_{\beta=1}^{k}\Bigg[\bigg(\frac{d}{dt}\frac{\partial T}{\partial\dot q_{s+\beta}}\bigg)\bigg(B_{s+\beta,\alpha}-\frac{\partial\dot q_{s+\beta}}{\partial\dot q_\alpha}\bigg)$$

$$- \frac{\partial T}{\partial \dot{q}_{s+\beta}} \left(\frac{d}{dt} \frac{\partial \dot{q}_{s+\beta}}{\partial \dot{q}_\alpha} - \frac{\partial \dot{q}_{s+\beta}}{\partial q_\alpha} - \sum_{\gamma=1}^{k} \frac{\partial \dot{q}_{s+\beta}}{\partial q_{s+\gamma}} B_{s+\gamma,\alpha} \right) \Bigg]_{\dot{q}_{s+\beta}=\dot{q}_{s+\beta}(q_1,\ldots,q_n,\dot{q}_1,\ldots,\dot{q}_s,t)}$$

$$= \tilde{Q}_\alpha \quad (\alpha = 1, \ldots, s). \tag{16}$$

This equation is not sufficient for the determination of the dynamics of the system because the number of the unknowns is larger than the number of determining equations. Therefore, one has to augment (16) with (2). Equation (16) is the main result valid in general. Below we demonstrate how special cases derives from this formula.

2.1 Conservative Systems

If the constraint system is conservative, i.e. if there exists a generalized potential

$$\begin{aligned} V &= V(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_s, \dot{q}_{s+1}(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_s, t), \ \ldots, \\ &\qquad \dot{q}_n(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_s, t), t) \\ &\equiv \tilde{V}(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_s, t), \end{aligned} \tag{17}$$

from which the generalized forces are derived

$$Q_\alpha = \frac{d}{dt} \frac{\partial V}{\partial \dot{q}_\alpha} - \frac{\partial V}{\partial q_\alpha} \quad (\alpha = 1, \ldots, n), \tag{18}$$

then (10) takes on the form

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\alpha} - \frac{\partial L}{\partial q_\alpha} + \sum_{\beta=1}^{k} \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_{s+\beta}} - \frac{\partial L}{\partial q_{s+\beta}} \right) B_{s+\beta,\alpha} = 0 \quad (\alpha = 1, \ldots, s). \tag{19}$$

Here we have introduced the Lagrangian $L = T - V = \tilde{T} - \tilde{V} = \tilde{L}$. Repeating the steps followed in (11–16), it is straightforward to derive the equations of motion for the conservative nonholonomic system, namely

$$\frac{d}{dt} \frac{\partial \tilde{L}}{\partial \dot{q}_\alpha} - \frac{\partial \tilde{L}}{\partial q_\alpha} - \frac{\partial \tilde{L}}{\partial q_{s+\beta}} B_{s+\beta,\alpha} + \sum_{\beta=1}^{k} \Bigg[\left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_{s+\beta}} \right) \left(B_{s+\beta,\alpha} - \frac{\partial \dot{q}_{s+\beta}}{\partial \dot{q}_\alpha} \right)$$

$$- \frac{\partial L}{\partial \dot{q}_{s+\beta}} \left(\frac{d}{dt} \frac{\partial \dot{q}_{s+\beta}}{\partial \dot{q}_\alpha} - \frac{\partial \dot{q}_{s+\beta}}{\partial q_\alpha} - \sum_{\gamma=1}^{k} \frac{\partial \dot{q}_{s+\beta}}{\partial q_{s+\gamma}} B_{s+\gamma,\alpha} \right) \Bigg]_{\dot{q}_{s+\beta}=\dot{q}_{s+\beta}(q_1,\ldots,q_n,\dot{q}_1,\ldots,\dot{q}_s,t)}$$

$$= 0 \quad (\alpha = 1, \ldots, s). \tag{20}$$

The left-hand side of (20) is the same as that of (16) except that the kinetic energy is replaced by the Lagrangian.

2.2 The Case of Cyclic Coordinates

In the event that $q_{s+1}, \ldots, q_n$ are cyclic coordinates, meaning that they do not appear in the expressions of $\tilde{T}$ and $\tilde{Q}_1, \ldots, \tilde{Q}_s$, then the relation (16) simplifies to

$$\frac{d}{dt}\frac{\partial \tilde{T}}{\partial \dot{q}_\alpha} - \frac{\partial \tilde{T}}{\partial q_\alpha} + \sum_{\beta=1}^{k}\left[\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_{s+\beta}}\right)\left(B_{s+\beta,\alpha} - \frac{\partial \dot{q}_{s+\beta}}{\partial \dot{q}_\alpha}\right)\right.$$
$$\left. - \frac{\partial T}{\partial \dot{q}_{s+\beta}}\left(\frac{d}{dt}\frac{\partial \dot{q}_{s+\beta}}{\partial \dot{q}_\alpha} - \frac{\partial \dot{q}_{s+\beta}}{\partial q_\alpha}\right)\right]_{\dot{q}_{s+\beta}=\dot{q}_{s+\beta}(q_1,\ldots,q_s,\dot{q}_1,\ldots,\dot{q}_s,t)}$$
$$= \tilde{Q}_\alpha \quad (\alpha = 1,\ldots,s). \tag{21}$$

For this special case, (21) solves the motion of $q_1,\ldots,q_s$ solely, and (2) solves $q_{s+1},\ldots,q_n$.

2.3 The Chetaev Case

If the constraint (1) is homogeneous in relative velocities, (6) is cast as [10]

$$B_{s+\beta,\alpha} = \frac{\partial \dot{q}_{s+\beta}}{\partial \dot{q}_\alpha} \quad (\beta = 1,\ldots,k). \tag{22}$$

Equation (16) becomes then

$$\frac{d}{dt}\frac{\partial \tilde{T}}{\partial \dot{q}_\alpha} - \frac{\partial \tilde{T}}{\partial q_\alpha} - \sum_{\beta=1}^{k}\frac{\partial \tilde{T}}{\partial q_{s+\beta}}\frac{\partial \dot{q}_{s+\beta}}{\partial \dot{q}_\alpha} - \sum_{\beta=1}^{k}\frac{\partial T}{\partial \dot{q}_{s+\beta}}\left(\frac{d}{dt}\frac{\partial \dot{q}_{s+\beta}}{\partial \dot{q}_\alpha}\right.$$
$$\left. - \frac{\partial \dot{q}_{s+\beta}}{\partial q_\alpha} - \sum_{\gamma=1}^{k}\frac{\partial \dot{q}_{s+\beta}}{\partial q_{s+\gamma}}\frac{\partial \dot{q}_{s+\gamma}}{\partial \dot{q}_\alpha}\right)_{\dot{q}_{s+\beta}=\dot{q}_{s+\beta}(q_1,\ldots,q_n,\dot{q}_1,\ldots,\dot{q}_s,t)}$$
$$= \tilde{Q}_\alpha \quad (\alpha = 1,\ldots,s). \tag{23}$$

This is just the equation of motion for the Chetaev system [14, 15].

2.4 The Voronets' Case

For many practical situations the constraints are linear in relative velocities, i.e.

$$\dot{q}_{s+\beta} = \sum_{\alpha=1}^{s} B_{s+\beta,\alpha}(q_1,\ldots,q_n,t)\dot{q}_\alpha + B_{s+\beta,t}(q_1,\ldots,q_n,t). \tag{24}$$

In this case (16) delivers

$$\frac{d}{dt}\frac{\partial \dot{q}_{s+\beta}}{\partial \dot{q}_\alpha} - \frac{\partial \dot{q}_{s+\beta}}{\partial q_\alpha} - \sum_{\gamma=1}^{k}\frac{\partial \dot{q}_{s+\beta}}{\partial q_{s+\gamma}}B_{s+\gamma,\alpha}$$
$$= \left(\sum_{\sigma=1}^{s}\frac{\partial B_{s+\beta,\alpha}}{\partial q_\sigma}\dot{q}_\sigma + \sum_{\gamma=1}^{k}\frac{\partial B_{s+\beta,\alpha}}{\partial q_{s+\gamma}}\dot{q}_{s+\gamma} + \frac{\partial B_{s+\beta,\alpha}}{\partial t}\right) - \left(\sum_{\sigma=1}^{s}\frac{\partial B_{s+\beta,\sigma}}{\partial q_\alpha}\dot{q}_\sigma + \frac{\partial B_{s+\beta,t}}{\partial q_\alpha}\right)$$
$$- \sum_{\gamma=1}^{k}\left(\sum_{\sigma=1}^{s}\frac{\partial B_{s+\beta,\sigma}}{\partial q_{s+\gamma}}\dot{q}_\sigma + \frac{\partial B_{s+\beta,t}}{\partial q_{s+\gamma}}\right)B_{s+\gamma,\alpha}$$
$$= \sum_{\sigma=1}^{s} A_{s+\beta,\alpha\sigma}\dot{q}_\sigma + A_{s+\beta,\alpha}, \tag{25}$$

where

$$A_{s+\beta,\alpha\sigma} = \frac{\partial B_{s+\beta,\alpha}}{\partial q_\sigma} - \frac{\partial B_{s+\beta,\sigma}}{\partial q_\alpha} + \sum_{\gamma=1}^{k}\left(\frac{\partial B_{s+\beta,\alpha}}{\partial q_{s+\gamma}} B_{s+\gamma,\sigma} - \frac{\partial B_{s+\beta,\sigma}}{\partial q_{s+\gamma}} B_{s+\gamma,\alpha}\right), \tag{26}$$

$$A_{s+\beta,\alpha} = \frac{\partial B_{s+\beta,\alpha}}{\partial t} - \frac{\partial B_{s+\beta,t}}{\partial q_\alpha} + \sum_{\gamma=1}^{k}\left(\frac{\partial B_{s+\beta,\alpha}}{\partial q_{s+\gamma}} B_{s+\gamma} - \frac{\partial B_{s+\beta,t}}{\partial q_{s+\gamma}} B_{s+\gamma,\alpha}\right). \tag{27}$$

Inserting (25) into (16) we retrieve the *Voronets' equation* [6, 16, 17]:

$$\begin{aligned} &\frac{d}{dt}\frac{\partial \tilde{T}}{\partial \dot{q}_\alpha} - \frac{\partial \tilde{T}}{\partial q_\alpha} - \sum_{\beta=1}^{k}\frac{\partial \tilde{T}}{\partial q_{s+\beta}} B_{s+\beta,\alpha} - \sum_{\beta=1}^{k}\frac{\partial T}{\partial \dot{q}_{s+\beta}}\left(\sum_{\sigma=1}^{s} A_{s+\beta,\alpha\sigma}\dot{q}_\sigma + A_{s+\beta,\alpha}\right) \\ &\quad = \tilde{Q}_\alpha \quad (\alpha = 1, \ldots, s). \end{aligned} \tag{28}$$

A special case of (24) is when the constraints are time-independent (scleronomic) and the coordinates $q_{s+1}, \ldots, q_n$ are cyclic, i.e.

$$B_{s+\beta,\alpha} = B_{s+\beta,\alpha}(q_1, \ldots, q_s), \qquad B_{s+\beta,t} = 0. \tag{29}$$

Equation (28) degenerates then to the *Chaplygin equation* [5, 14]:

$$\frac{d}{dt}\frac{\partial \tilde{T}}{\partial \dot{q}_\alpha} - \frac{\partial \tilde{T}}{\partial q_\alpha} - \sum_{\beta=1}^{k}\frac{\partial T}{\partial \dot{q}_{s+\beta}}\left(\frac{\partial B_{s+\beta,\alpha}}{\partial q_\sigma} - \frac{\partial B_{s+\beta,\sigma}}{\partial q_\alpha}\right)\dot{q}_\sigma = \tilde{Q}_\alpha \quad (\alpha = 1, \ldots, s). \tag{30}$$

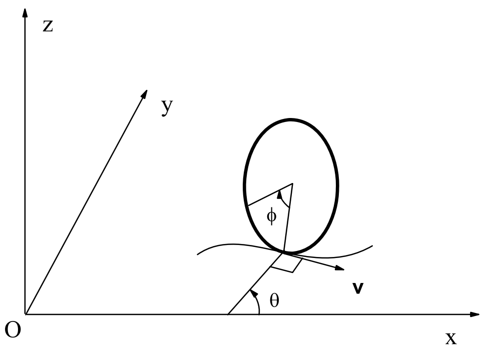
3 An Illustrative Example

For a demonstration we consider deliberately a simple example, namely that of a vertical uniform disk rolling on a horizontal xy-plane without slipping (cf. Fig. 1). The radius and the mass of the disk are respectively a and m. We may choose the generalized coordinates to be the Cartesian coordinates x, y of the disk center, the angle of rotation ϕ with respect the disk axis, and an angle θ between the axis of disk and the x-axis. The constraints in the directions tangential and normal to the trace of the contact point read respectively

$$\dot{x}\sin\theta - \dot{y}\cos\theta - a\dot{\phi} = 0, \tag{31}$$

$$\dot{x}\cos\theta + \dot{y}\sin\theta = 0. \tag{32}$$

Fig. 1 A vertical disk rolling on a horizontal plane



We may choose $\dot{q}_1 = \dot{\phi}$, $\dot{q}_2 = \dot{\theta}$ as the independent velocities, and $\dot{q}_3 = \dot{x}$, $\dot{q}_4 = \dot{y}$ as the dependent velocities. From (31) and (32), we can solve

$$\dot{q}_3 = \dot{x} = a\dot{\phi}\sin\theta, \tag{33}$$

$$\dot{q}_4 = \dot{y} = -a\dot{\phi}\cos\theta, \tag{34}$$

which implies that (cf. (5))

$$B_{31} = a\sin\theta, \qquad B_{32} = 0, \tag{35}$$

$$B_{42} = -a\cos\theta, \qquad B_{42} = 0. \tag{36}$$

The kinetic energy of the disk reads then

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}(I_\phi\dot{\phi}^2 + I_\theta\dot{\theta}^2), \tag{37}$$

where $I_\phi = ma^2/2$ and $I_\theta = ma^2/4$ are respectively the momenta of inertia around the disk axis and around the vertical diameter of the disk. Inserting (33) and (34) into (37) yields

$$\tilde{T} = \frac{1}{2}(I_\phi + ma^2)\dot{\phi}^2 + \frac{1}{2}I_\theta\dot{\theta}^2. \tag{38}$$

Inserting (37), (38), (33), and (34) into (16) we obtain the equations for ϕ and θ in the form

$$\begin{aligned} &\frac{d}{dt}\big[\big(I_\phi + ma^2\big)\dot{\phi}\big] - 0 - 0 + \left\{\left[\frac{d}{dt}(m\dot{x})\right]\cdot 0 - m\dot{x}\left[\frac{d}{dt}(a\sin\theta) - 0 - 0\right]\right. \\ &\qquad \left.\left. + \left[\frac{d}{dt}(m\dot{y})\right]\cdot 0 - m\dot{y}\left[\frac{d}{dt}(-a\cos\theta) - 0 - 0\right]\right\}\right|_{\dot{x}=a\dot{\phi}\sin\theta,\, \dot{y}=-a\dot{\phi}\cos\theta} = 0, \end{aligned} \tag{39}$$

$$\begin{aligned} &\frac{d}{dt}(I_\theta\dot{\theta}) - 0 - 0 + \left\{\left[\frac{d}{dt}(m\dot{x})\right]\cdot 0 - m\dot{x}[0 - a\dot{\phi}\cos\theta - 0] + \left[\frac{d}{dt}(m\dot{y})\right]\cdot 0\right. \\ &\qquad \left.\left. - m\dot{y}[0 - a\dot{\phi}\sin\theta - 0]\right\}\right|_{\dot{x}=a\dot{\phi}\sin\theta,\, \dot{y}=-a\dot{\phi}\cos\theta} = 0. \end{aligned} \tag{40}$$

A straightforward simplifications lead us to the conclusion that

$$\ddot{\phi} = 0, \tag{41}$$

$$\ddot{\theta} = 0. \tag{42}$$

4 Conclusions and Final Remarks

Based on the velocity-determined virtual displacement condition we presented a new approach to the treatment of the classical motion of systems with nonholonomic constraints. General, multiplier-free equations of motion in the Lagrangian form are derived. Established results derive from our scheme as special cases: our results reduce to the equations of the Chetaev and Voronets' cases if the constraints are homogeneous or linear in relative velocities. The method is illustrated on a simple well-known example. Ongoing research is focused on the development of a Hamiltonian formulation [1] and a geometric formulation [2] based on the velocity-determined virtual displacements.

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